Solutions to Stochastic Differential Equations by Øksendal

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Contents

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2. Some Mathematical Preliminaries

Exercise 2.1

Suppose that $X : \Omega \to \mathbb{R}$ is a function which takes only countably many values $a_1, a_2, \ldots \in \mathbb{R}$.

(a) Show that X is a random variable if and only if

$$X^{-1}(a_i) \in \mathcal{F} \text{ for all } i \in \mathbb{N}.$$

(b) Suppose that X is a random variable. Show that

$$E[|X|] = \sum_{i=1}^{\infty} |a_i| P(X = a_i).$$

(c) If X is a random variable and $E[|X|] < \infty$, show that

$$E[X] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

(d) If X is a random variable and $f : \mathbb{R} \to \mathbb{R}$ is measurable and bounded, show that

$$E[f(X)] = \sum_{i=1}^{\infty} f(a_i) P(X = a_i).$$

Solution.

For (a), suppose first that X is a random variable. Since $\{a_i\}$ are Borel sets, $X^{-1}(a_i) \in \mathcal{F}$ for all $i \in \mathbb{N}$. Conversely, assume that $X^{-1}(a_i) \in \mathcal{F}$ for all a_i. Since the range of X is $\{a_i\}_{i\in\mathbb{N}}$, for any Borel set $B \subset \mathbb{R}$, $X^{-1}(B) = \bigcup_{a_i \in B} X^{-1}(a_i) \in \mathcal{F}$, by the definition of σ-algebra. Thus, X is a random variable.

For (b), since X takes only countably many values, so does $|X|$ with $\{|a_i|\}_{i\in\mathbb{N}}$. By the definition of expectation, we have

$$E[|X|] = \sum_{i=1}^{\infty} |a_i| P(X = a_i)$$

in the extended sense.

For (c), since $E[|X|] < \infty$ and X is a random variable, the series converges absolutely and is well-defined. Hence

$$E[X] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

For (d), since f is measurable, $f^{-1}(B)$ is Borel and $X^{-1}f^{-1}(B)$ is measurable. $f(X)$ takes only countably many values, $f(a_1), f(a_2), \ldots$. The definition of expectation gives us

$$E[f(X)] = \sum_{i=1}^{\infty} f(a_i)P(f(X) = f(a_i)) = \sum_{i=1}^{\infty} f(a_i)P(X = a_i).$$

■

Exercise 2.2

$X : \Omega \to \mathbb{R}$ is a random variable. The distribution function F of X is defined as

$$F(x) = P(X \le x).$$

(a) Prove that F has the following properties:

- *(i) $0 \le F \le 1$, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.*
- *(ii) F is non-decreasing.*
- *(iii) F is right-continuous.*

(b) $g : \mathbb{R} \to \mathbb{R}$ is measurable such that $E[|g(X)|] < \infty$. Show that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)dF(x).$$

(c) Let $p(x) \ge 0$ be measurable on $\mathbb{R}$ be the density of X, i.e.,

$$F(x) = \int_{-\infty}^{x} p(t)dt.$$

Find density of B_t^2.

Solution.

For (a), since P is a probability measure, $0 \le P(S) \le 1$ for any $S \in \mathcal{F}$. In particular, $0 \le P(X \le x) \le 1$ for all $x \in \mathbb{R}$. Also, we can take $x_n \searrow -\infty$ and $|X \le x_n| \searrow \emptyset$ as $n \to \infty$. Hence

$$\lim_{x\to-\infty} F(x) = \lim_{n\to\infty} P(X \le x_n) = P(\emptyset) = 0.$$

Similarly, we can take $x_n \nearrow \infty$ and $|X \le x_n| \nearrow \Omega$ as $n \to \infty$. Hence

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} P(X \le x_n) = P(\Omega) = 1.$$

(i) is proved. For (ii), F is non-decreasing because if $x_1 < x_2$, then

$$F(x_1) = P(X \le x_1) \le P(X \le x_2) = F(x_2).$$

For (iii), let $h > 0$.

$$F(x + h) - F(x) = P(X \le x + h) - P(X \le x) = P(x < X \le x + h).$$

For any $y > x$, there exists $h > 0$ such that $y > x + h$. Thus $(x, x+h] \searrow \varnothing$ as $h \to 0$. Hence

$$F(x+h) - F(x) = P(x < X \leq x+h) \to P(\varnothing) = 0$$

as $h \to 0$. Therefore, F is right-continuous.

For (b), by definition of expectation, the left-hand side is

$$E[g(X)] = \int_{\mathbb{R}} g(x) d\mu_X(x),$$

where $\mu_X(B) = P(X^{-1}(B))$ for any Borel set $B \subset \mathbb{R}$.

For (c),

$$F(x) = P(B_t^2 \leq x) = P(B_t \leq \sqrt{x}) = \int_{-\infty}^{\sqrt{x}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) du.$$

Hence,

$$p(u) = \frac{d}{dx} F(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x}{2t}\right) \frac{1}{2\sqrt{x}}.$$

■

Exercise 2.3

Let $\{\mathcal{F}_i\}_{i \in \mathcal{I}}$ be a collection of σ-algebras on Ω. Prove that

$$\mathcal{F} = \bigcap_{i \in \mathcal{I}} \mathcal{F}_i$$

is again a σ-algebra.

Solution.

First, since $\mathcal{F}_i$ are σ-algebras, they contain $\varnothing$ and hence $\varnothing \in \mathcal{F}$. For any $A \in \mathcal{F}$, $A \in \mathcal{F}_i$ for all $i \in \mathcal{I}$ and hence $A^c \in \mathcal{F}_i$ for all $i \in \mathcal{I}$. Thus $A^c \in \mathcal{F}$. Finally, for any countable collection $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$, we have $A_n \in \mathcal{F}_i$ for all $i \in \mathcal{I}$ and all $n \in \mathbb{N}$. Then $\bigcup_n A_n \in \mathcal{F}_i$ for all $i \in \mathcal{I}$. Hence $\bigcup_n A_n \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a σ-algebra. ■

Exercise 2.4

(a) Let $X : \Omega \to \mathbb{R}$ be a random variable such that $E[|X|^p] < \infty$ for some $p \in (0, \infty)$. Prove the Chebyshev's inequality:

$$P(|X| \geq \lambda) \leq \frac{1}{\lambda^p} E[|X|^p]$$

for any $\lambda > 0$.

(b) Suppose there exists $k > 0$ such that $M = E[\exp(k|X|)] < \infty$. Prove that $P(|X| \geq \lambda) \leq Me^{-k\lambda}$ for any $\lambda > 0$.

Solution.

For (a), directly estimate that

$$P(|X| \geq \lambda) = \int_{\Omega} \chi_{\{|X|^p \geq \lambda^p\}} dP \leq \int_{\Omega} \frac{|X|^p}{\lambda^p} dP = \frac{1}{\lambda^p} E\left[|X|^p\right].$$

(b) is similar:

$$P(|X| \geq \lambda) = \int_{\Omega} \chi_{\{\exp(k|X|) \geq \exp(k\lambda)\}} dP \leq \int_{\Omega} \exp(k\,|X|) \exp(-k\lambda) dP = M \exp(-k\lambda).$$

■

Exercise 2.5
Let $X, Y : \Omega \to \mathbb{R}$ be two independent random variables and assume for simplicity that X, Y are bounded. Prove that

$$E[XY] = E[X]\,E[Y].$$

Solution.

For any $\epsilon > 0$, by definition of the expectation, we can find simple functions s and t on Ω such that

$$\int |s - X|\, dP < \epsilon, \quad \int |t - Y|\, dP < \epsilon, \quad \Rightarrow \quad \left|E[X] - \int s dP\right| < \epsilon, \quad \left|E[Y] - \int t dP\right| < \epsilon,$$

where s and t can be written as

$$s = \sum_{i=1}^{n} s_i \chi_{X^{-1}[s_i, s_{i+1})} \quad \text{and} \quad t = \sum_{j=1}^{m} t_j \chi_{Y^{-1}[t_j, t_{j+1})},$$

with s_i and t_j being arranged in ascending order. Thus,

$$\begin{aligned}
\int st dP &= \sum_{i=1}^{n} \sum_{j=1}^{m} s_i t_j P(\{X \in [s_i, s_{i+1})\} \cap \{Y \in [t_j, t_{j+1})\}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} s_i t_j P(X \in [s_i, s_{i+1})) P(Y \in [t_j, t_{j+1})) \\
&= \left(\sum_{i=1}^{n} s_i P(X \in [s_i, s_{i+1}))\right) \left(\sum_{j=1}^{m} t_j P(Y \in [t_j, t_{j+1}))\right) = \left(\int s dP\right) \left(\int t dP\right).
\end{aligned}$$

Also,

$$\left|E[XY] - \int st dP\right| \leq \left|\int |X - s|\,|t|\, dP\right| + \left|\int |Y - t|\,|X|\, dP\right|.$$

X and Y are bounded, say by M and N respectively. Then t is also bounded by N from our construction. Thus

$$\left|E[XY] - \int st dP\right| \leq M\epsilon + N\epsilon.$$

Combine the results above, we arrive at

$$
\begin{aligned}
|E[XY] - E[X]E[Y]| &\le \left|E[XY] - \int stdP\right| + \left|E[X]E[Y] - \int sdP \int tdP\right| \\
&\le (M+N)\epsilon + \left|E[X] - \int sdP\right| \left|\int tdP\right| + \left|E[Y] - \int tdP\right| |E[X]| \\
&\le (M+N)\epsilon + \epsilon N + \epsilon M.
\end{aligned}
$$

Since ϵ is arbitrary, we conclude that $E[XY] = E[X]E[Y]$. ■

Exercise 2.6

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, \ldots \in \mathcal{F}$ be sets such that

$$\sum_{i=1}^{\infty} P(A_i) < \infty.$$

Prove the Borel-Cantelli lemma:

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i\right) = 0.$$

Solution.

Set $B_m = \bigcup_{i=m}^{\infty} A_i$ be measurable. Then

$$P(B_m) \le \sum_{i=m}^{\infty} P(A_i) \longrightarrow 0$$

as $m \longrightarrow \infty$ by the assumption. Thus

$$P\left(\bigcap_{m=1}^{\infty} B_m\right) \le \lim_{n\to\infty} P\left(\bigcap_{m=1}^{n} B_m\right) \le \lim_{n\to\infty} P(B_n) = 0.$$

■

Exercise 2.7

(a) Suppose $G_1, \ldots, G_n$ are disjoint sets in $\mathcal{F}$ such that $\bigcup_{i=1}^{n} G_i = \Omega$. Prove that the family

$$\mathcal{G} = \{G \mid G \text{ is a union of some } G_i\} \cup \{\varnothing\}$$

is a σ-algebra.

(b) Prove that every finite σ-algebra is of type $\mathcal{G}$ as in (a).

(c) Let $\mathcal{F}$ be a finite σ-algebra on Ω and $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Prove that X is simple.

Solution.

For (a), first, $\emptyset \in \mathcal{G}$ by definition. Let $G \in \mathcal{G}$. Then $G = \cup_{i \in \mathcal{I}} G_i$ for some $\mathcal{I} \subset \{1, \ldots, n\}$, with the convention that $\cup_{i \in \emptyset} G_i = \emptyset$. Then $G^c = \bigcup_{i \notin \mathcal{I}} G_i \in \mathcal{G}$. Lastly, for countably many $G_i \in \mathcal{G}$, since $\mathcal{G}$ is finite, there are in fact finitely many distinct G_i and the union must lie in $\mathcal{G}$ by the definition. Hence $\mathcal{G}$ is a σ-algebra.

For (b), let $\mathcal{F}$ be a finite σ-algebra. Consider the collection

$$\mathcal{S} = \{S \in \mathcal{F} \mid S \cap F = \emptyset \text{ or } S \text{ for all } F \in \mathcal{F}\} .$$

Since $\mathcal{F}$ is finite, $\mathcal{S}$ is also finite. We first check that every distinct sets in $\mathcal{S}$ are disjoint. Suppose not. There are $S_1, S_2 \in \mathcal{S}$ such that $S_1 \cap S_2$ is non-empty. Then $S_1 \cap S_2 = S_1 = S_2$, contradicting the assumption that S_1 and S_2 are distinct. Thus every distinct sets in $\mathcal{S}$ are disjoint. Next, we check that $\bigcup_{S \in \mathcal{S}} S = \Omega$. If not, let $A = \Omega \setminus \bigcup_{S \in \mathcal{S}} S$ be non-empty and $A \cap F$ is a non-empty proper subset of A for some $F \in \mathcal{F}$. But then $A \cap F$ or $A \cap F^c$ must satisfy the condition that there is some $F' \in \mathcal{F}$ such that $A \cap F \cap F'$ or $A \cap F^c \cap F'$ is non-empty, proper subset of $A \cap F$ or $A \cap F^c$ respectively. Note that $F' \neq F$ and the process continues. In the end, we can find a infinite sequence of distinct sets lying in $\mathcal{F}$, contradicting the finiteness of $\mathcal{F}$. Thus $\bigcup_{S \in \mathcal{S}} S = \Omega$. Finally, by (a),

$$\mathcal{G} = \{G \mid G \text{ is a union of some } S \in \mathcal{S}\} \cup \{\emptyset\}$$

is a σ-algebra. It remains to show that $\mathcal{G} = \mathcal{F}$. Clearly, $\mathcal{G} \subset \mathcal{F}$ since $\mathcal{S} \subset \mathcal{F}$. For any $F \in \mathcal{F}$, we can write $F = \bigcup_{i=1}^{n} S_i$ for some $S_i \in \mathcal{S}$. Thus $F \in \mathcal{G}$. We end up with $\mathcal{G} = \mathcal{F}$.

For (c), suppose that X can take infinitely many values $\{a_i\}_{i \in \mathcal{I}}$. Since X is $\mathcal{F}$-measurable, $X^{-1}(\{a_i\}) \in \mathcal{F}$ for all $i \in \mathcal{I}$. In particular, $X^{-1}(\{a_i\})$ and $X^{-1}(\{a_j\})$ are disjoint for all $i \neq j$. This implies that $\mathcal{F}$ contains infinitely many disjoint sets, contradicting the finiteness of $\mathcal{F}$. Thus X can only take finitely many values and is simple. ■

Exercise 2.8

Let B_t be Brownian motion on $\mathbb{R}$, $B_0 = 0$. Put $E = E^0$.